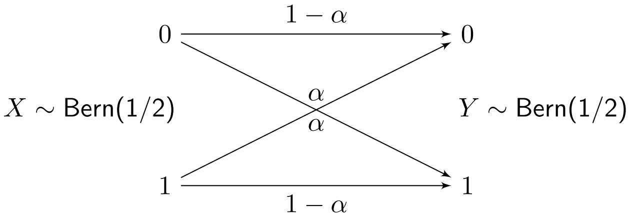
Hypercontractivity, Maximal Correlation and Non-cooperative Simulation

Zi Yin and Youngsuk Park

Example1

DSBS(Doubly Symmetric Binary Source):

- **Model**:



Example1

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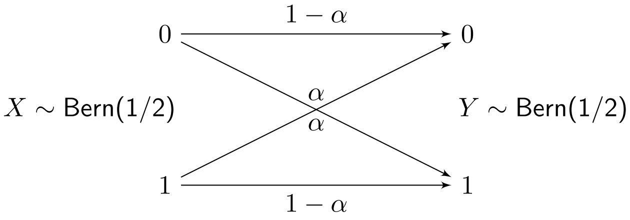
- **Model**:



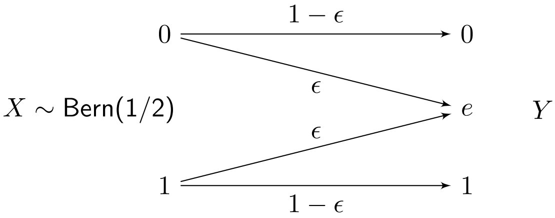
- **Common Information Measures**:

$$I(X;Y) = 1 - H(\alpha)$$
$$\rho_m^2(X;Y) = s^*(X;Y) = (1-2\alpha)^2$$

Example 2

SBES(Symmetric Binary Erasure Source):

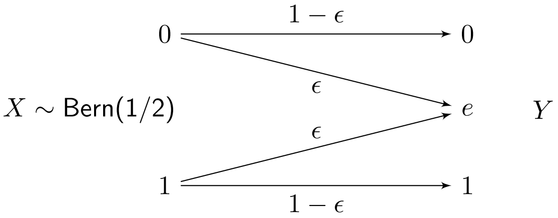
- **Model**:



Example 2

SBES(Symmetric Binary Erasure Source):

- **Model**:

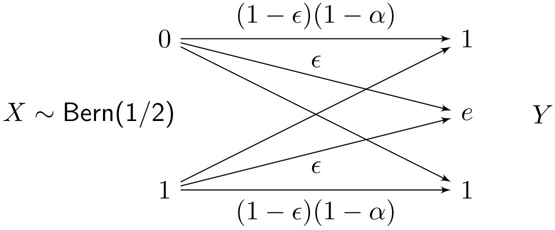


- **Common Information Measures**:

$$I(X;Y) = \rho_m^2(X;Y) = s^*(X;Y) = 1-\epsilon$$

BSBES(Binary Symmetric and Binary Erasure Source):

- **Model**:



BSBES(Binary Symmetric and Binary Erasure Source):

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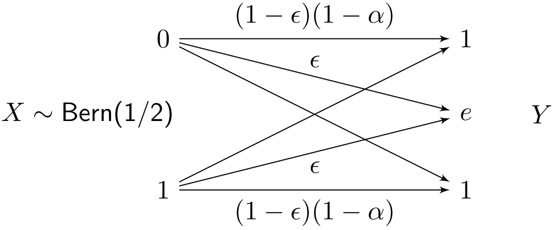

- **Common Information Measures**:

$$I(X;Y) = (1-\epsilon)(1-H(\alpha))$$
$$\rho_m^2(X;Y) = s^*(X;Y) = (1-\epsilon)(1-2\alpha)^2$$

- **Remind** Why do we care two common inforamation meausres?

Non-Interactive Simulation

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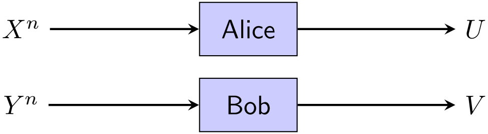
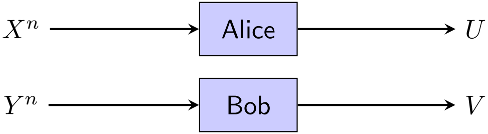
- **Scenario: Non-Interactive Simulation**



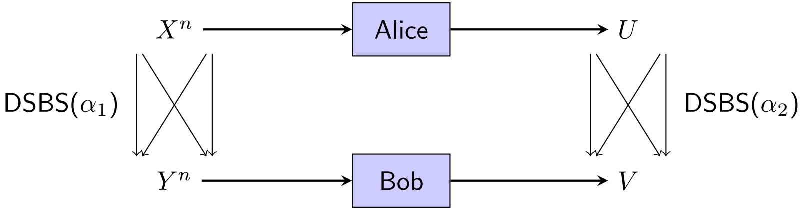
Non-Interactive Simulation

- **Remind** Why do we care two common inforamation meausres?
- **Scenario: Non-Interactive Simulation**



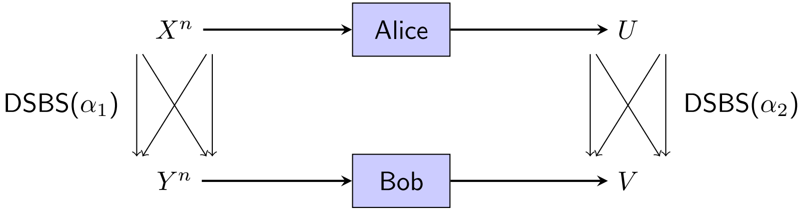
- **Necessary condition**: By using data processing inequality and tesorized property of ρ_m, r_p
  - $\rho(X;Y) \geq \rho(U,V)$
  - $r_p(X;Y) \geq r_p(U,V)$ for all p

Non-Interactive Simulation: Example 1



- Necessary Condition:
  $\rho_m(X;Y) \geq \rho_m(U;V) \Leftrightarrow |1-2\alpha_1| \geq |1-2\alpha_2| \Leftrightarrow \alpha_1 \leq \alpha_2$

Non-Interactive Simulation: Example 1



- Sufficient Condition: $\alpha_1 \leq \alpha_2$.

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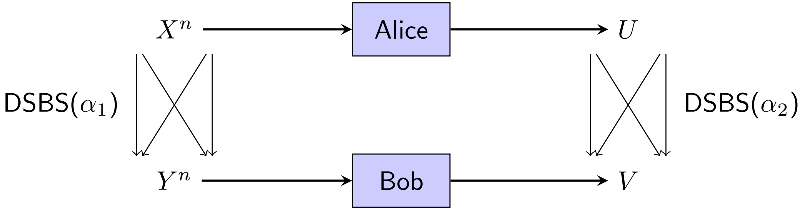

- Sufficient Condition: $\alpha_1 \leq \alpha_2$. How?

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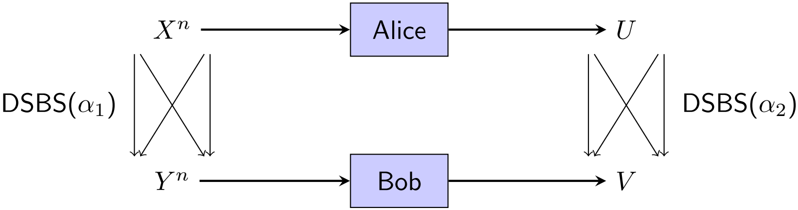

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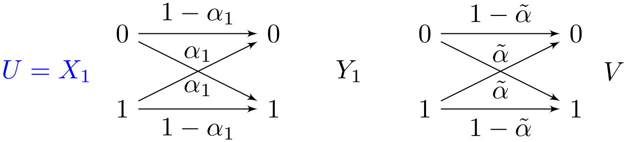

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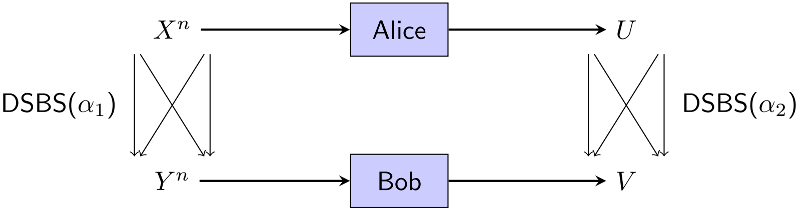

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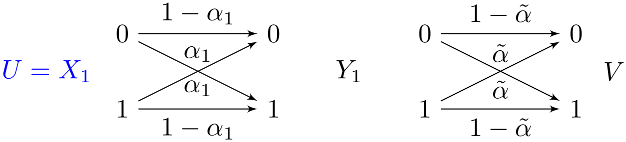
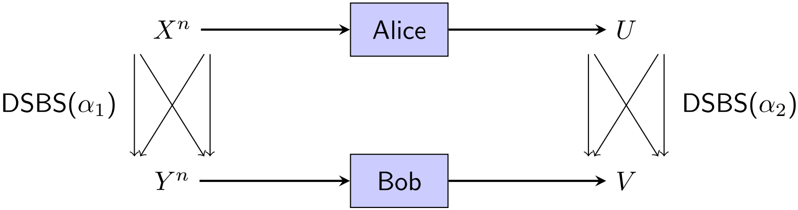


- Can we make such virtual channel ?



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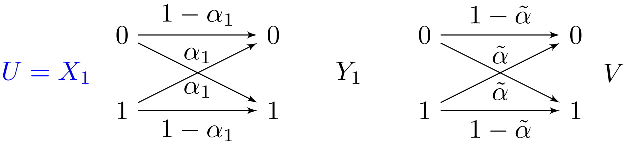

- Can we make such virtual channel ? Yes!

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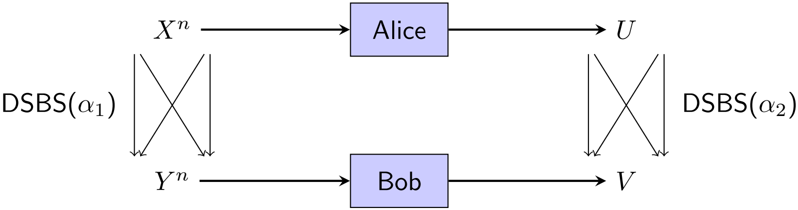
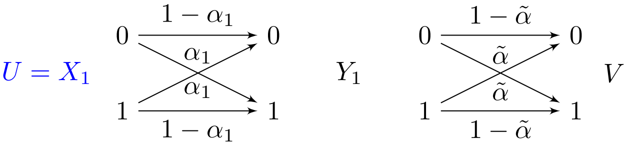
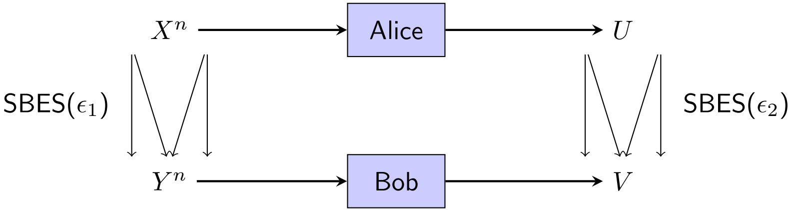


- Sufficient Condition: $\alpha_1 \leq \alpha_2$. How?



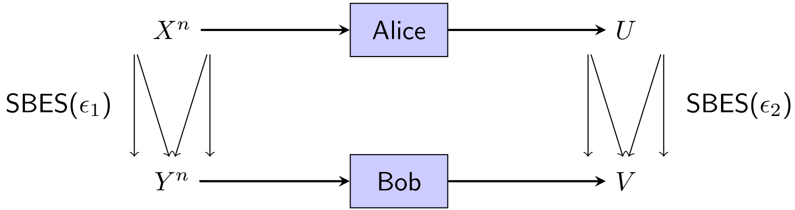
- Can we make such virtual channel ? Yes!
  For example, given a $\tilde{\alpha}$, there eixst n, K_n s.t $\mathbb{P}(S_n(Y^n) < K_n) \approx \tilde{\alpha}$.
  Thus $Z|\{Y_1\} = Y_1 \oplus 1$ if $S_n(Y^n) < K_n$, $Z|\{Y_1\} = Y_1$ otherwise.

Non-Interactive Simulation: Example 2



- Necessary Condition:
  $\rho_m(X;Y) \geq \rho_m(U;V) \Leftrightarrow 1-\epsilon_1 \geq 1-\epsilon_2 \Leftrightarrow \epsilon_1 \leq \epsilon_2$

Non-Interactive Simulation: Example 2



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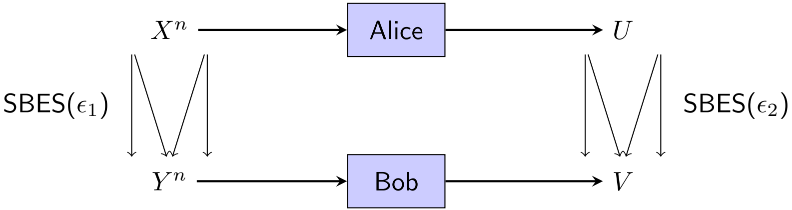

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  - How? Choose $\tilde{\epsilon} = \epsilon_2 - \epsilon_1$

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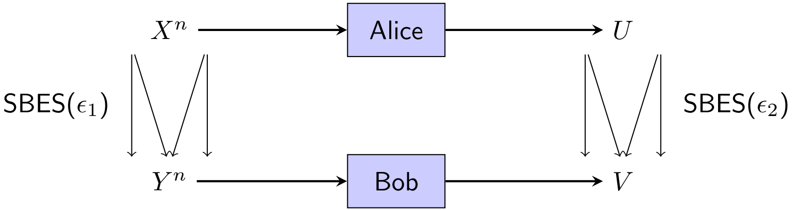

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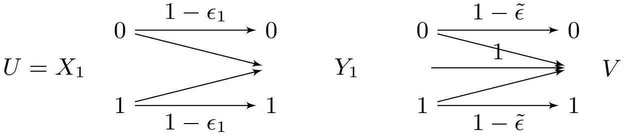

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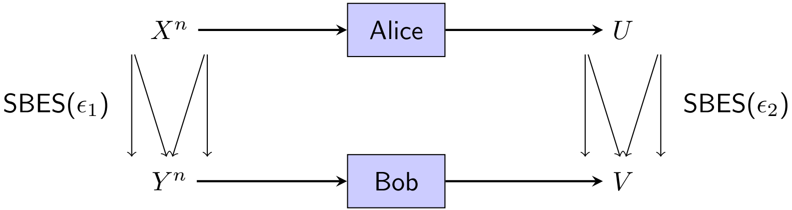

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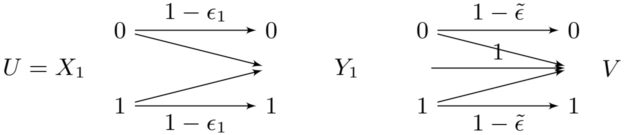
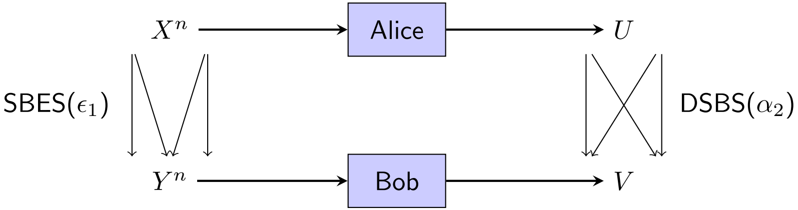


  - Bob makes additional erasure by observing Y^n, which consequently occur with probability $\tilde{\epsilon}$

Non-Interactive Simulation:Example 3



- Necessary Condition:

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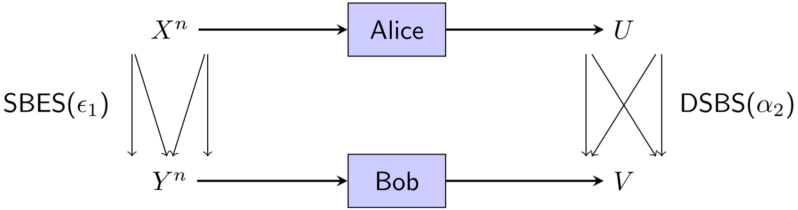
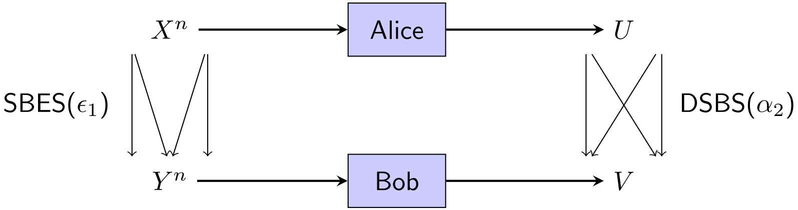


- Necessary Condition:
  - $\rho_m^2(X;Y) \geq \rho_m^2(U;V) \Leftrightarrow 1-\epsilon_1 \geq (1-2\alpha_2)^2 \Leftrightarrow \epsilon_1 \leq 4\alpha_2(1-\alpha_2)$

Non-Interactive Simulation:Example 3



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  - Match the channel condition of less noisy DM-BC

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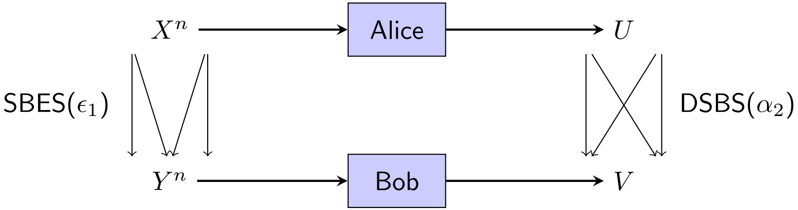

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  - Match the channel condition of less noisy DM-BC
- Sufficient Condition: $\epsilon_1 \leq 2\alpha_2$

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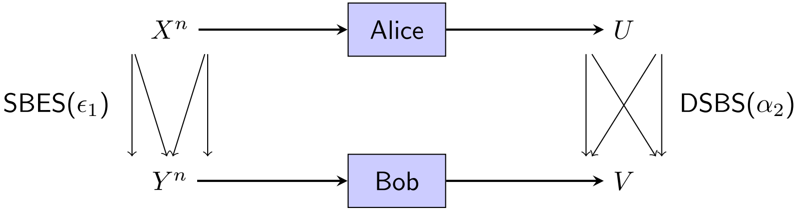

- Necessary Condition:
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  - Match the channel condition of less noisy DM-BC
- Sufficient Condition: $\epsilon_1 \leq 2\alpha_2$
  - Match the channel condition of degraded DM-BC.

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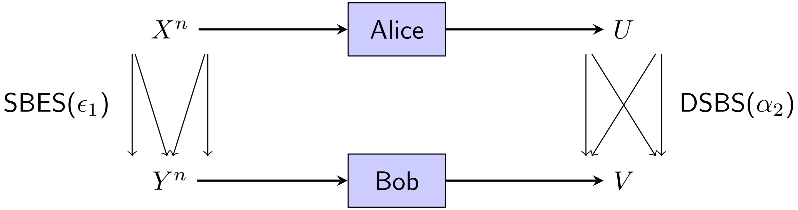

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  - Match the channel condition of less noisy DM-BC
- Sufficient Condition: $\epsilon_1 \leq 2\alpha_2$
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- What if $2\alpha_2 \leq \epsilon_1 \leq \epsilon_1 \leq 4\alpha_2(1-\alpha_2)$?

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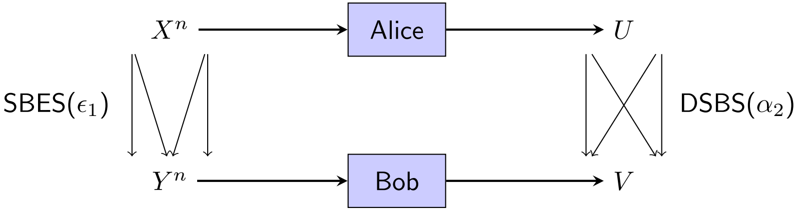

- Necessary Condition:
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  - Match the channel condition of less noisy DM-BC
- Sufficient Condition: $\epsilon_1 \leq 2\alpha_2$
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- What if $2\alpha_2 \leq \epsilon_1 \leq \epsilon_1 \leq 4\alpha_2(1 - \alpha_2)$? Let's see the achievability first.

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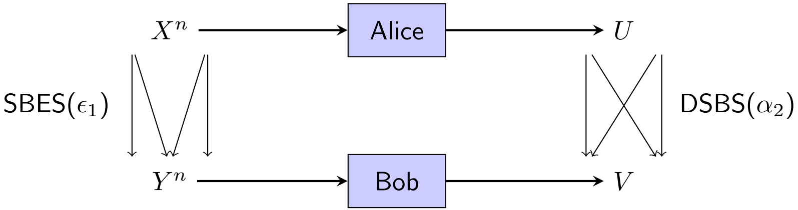
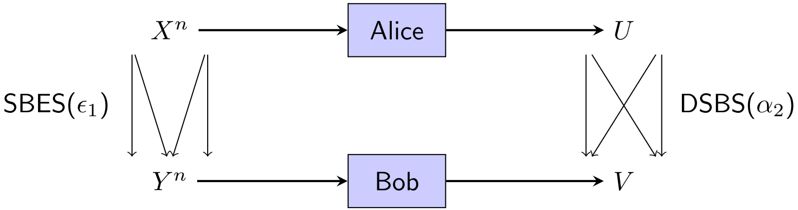
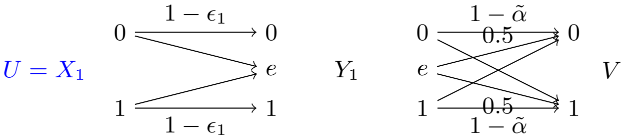


- Sufficient Condition: $\epsilon_1 \leq 2\alpha_2$



- Sufficient Condition: $\epsilon_1 \leq 2\alpha_2$
  - How? Choose $\tilde{\alpha}$ satisfying $(1-\epsilon_1)(\tilde{\alpha}) = \alpha$



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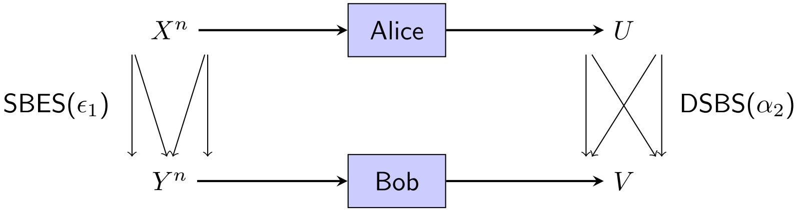

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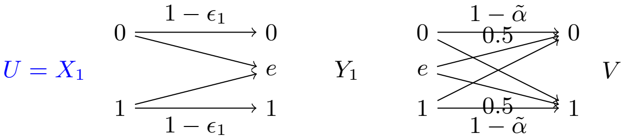
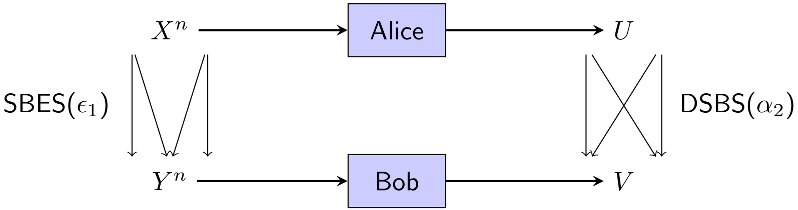


- What if $2\alpha_2 \leq \epsilon_1$?

Non-Interactive Simulation:Example 3



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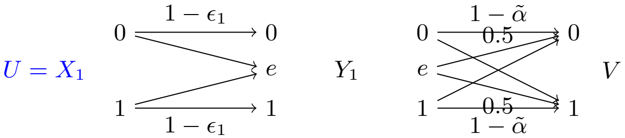
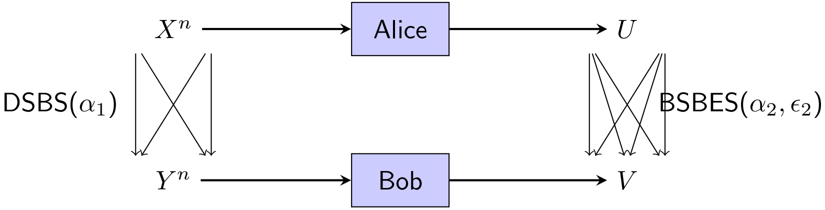


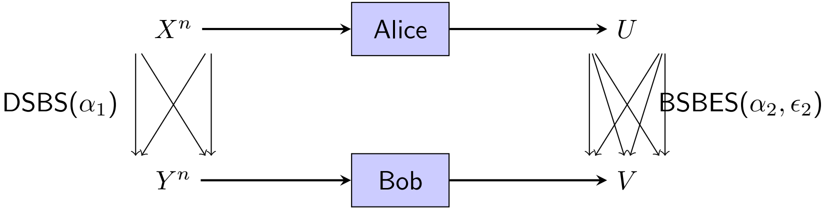
- What if $2\alpha_2 \leq \epsilon_1$? We don't know
  (because at least one of crossover probability can be greater than α_2.)

Non-Interactive Simulation:Example 4



- Necessary Condition:
  $\rho_m(X;Y) \geq \rho_m(U;V) \Leftrightarrow (1-\alpha_1)^2 \geq (1-\epsilon_2)(1-2\alpha_2)^2$

Non-Interactive Simulation:Example 4



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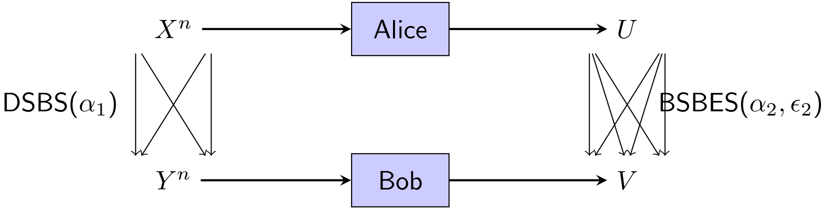

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- Sufficient Condition:
  - $\alpha_1 \leq \alpha_2 \Leftrightarrow (1-2\alpha_1)^2 \geq (1-2\alpha_2)^2$
  - How? Next slide.

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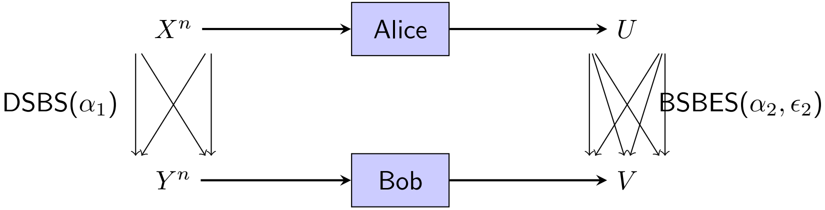

- Necessary Condition:
  $\rho_m(X;Y) \geq \rho_m(U;V) \Leftrightarrow (1-\alpha_1)^2 \geq (1-\epsilon_2)(1-2\alpha_2)^2$
- Sufficient Condition:
  - $\alpha_1 \leq \alpha_2 \Leftrightarrow (1-2\alpha_1)^2 \geq (1-2\alpha_2)^2$
  - How? Next slide.
- What about the $(1-\epsilon_2)(1-2\alpha_2)^2 \leq (1-2\alpha_1)^2 \leq (1-2\alpha_2)^2$?

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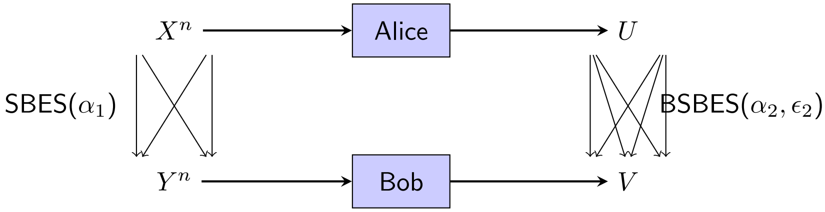

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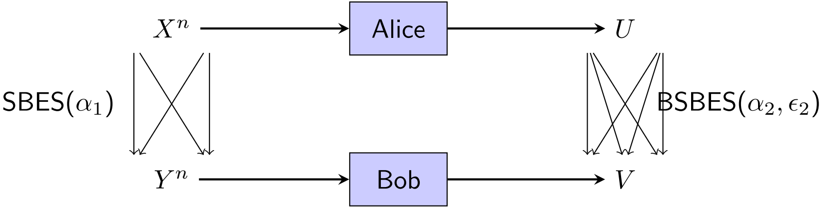

- Sufficient Condition: $\alpha_1 \leq \alpha_2$
- How? Both Alice and Bob do their jobs.

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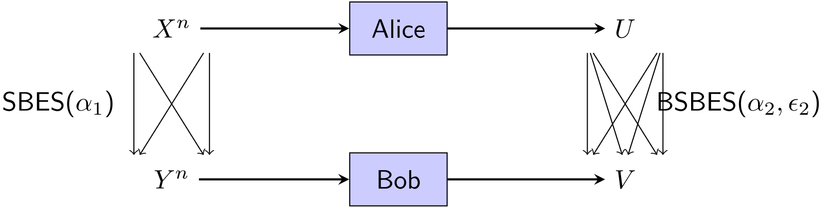

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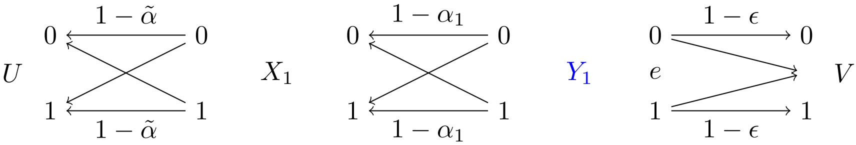

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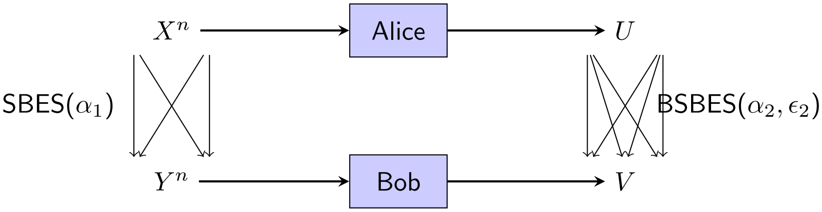

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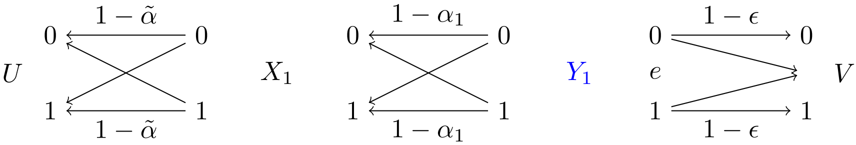

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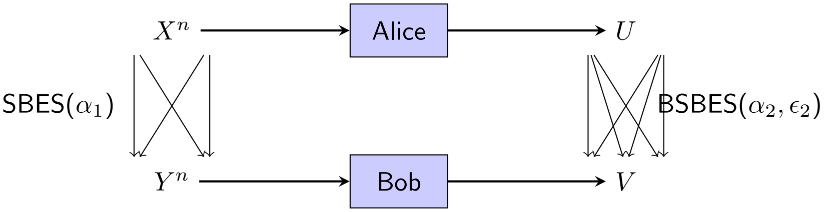

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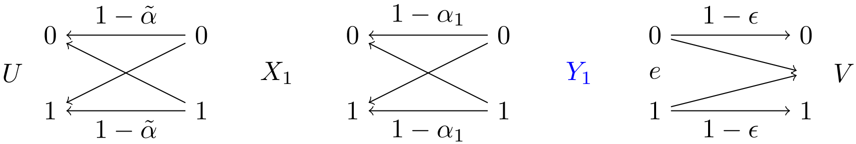

- For above virtual channel, U, V is BSBES
- In previous example, we showed that Both can generate BEC and Alice can also generate BSC.

Non-Interactive Simulation:Example 5

- Note in the previous 4 examples, ρ_m and s^* are the same, hence they always gave same necessary condition..
  Question: Are there are cases where one is more powerful?

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  Question: Are there are cases where one is more powerful?

- Answer (Kamath '12): Yes, consider $(X, Y)$is DSBS(α) and Where we want to simulate $(U, V) \sim p(u, v)$ binary, with

$$\mathbb{P}_{U,V}(0,0) = \mathbb{P}_{U,V}(0,1) = \mathbb{P}_{U,V}(1,0) = \frac{1}{3}$$

Non-Interactive Simulation:Example 5

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$$\rho_m(X;Y) = 1 - 2\alpha \geq \frac{1}{2} = \rho_m(U,V)$$

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- Then hypercontractivity gives

$$Pr\{\tilde{U} = 1, \tilde{V} = 1\} \geq Pr\{\tilde{U} = 1\}^{\frac{1}{2\alpha}} Pr\{\tilde{V} = 1\}^{\frac{1}{2\alpha}},$$

  Clearly the LHS is bounded away from zero, a contradiction!

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